# Constraints to a Yukawa gravitational potential from laser data to LAGEOS satellites

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## Abstract

In this paper we investigate the possibility of constraining the hypothesis of a fifth force at the length scale of two Earth's radii by investigating the effects of a Yukawa gravitational potential on the orbits of the laser–ranged LAGEOS and LAGEOS II satellites. The existing constraints on the Yukawa coupling $\alpha$, obtained by fitting the LAGEOS orbit, are of the order of $|\alpha| < 10^{-5} - 10^{-8}$ for distances of the order of $10^9$ cm. Here we show that with a suitable combination of LAGEOS and LAGEOS II data it should be possible to constrain $\alpha$ at a level of $4 \times 10^{-12}$ or less. Various sources of systematic errors are accounted for, as well. Their total impact amounts to $1 \times 10^{-11}$ during an observational time span of 5 years. In the near future, when the new data on the terrestrial gravitational field will be available from the CHAMP and GRACE missions, these limits will be further improved. The use of the proposed LARES laser–ranged satellite would yield an experimental accuracy in constraining $\alpha$ of the order of $1 \times 10^{-12}$.

# 1  Introduction

In this paper we investigate the constraints that can be posed on the existence of a possible fifth force by the analysis of the laser–ranged data to LAGEOS II and LAGEOS Earth satellites. We will consider a potential energy including a Yukawa term of the form [*Ohanian and Ruffini*, 1994; *Ciufolini and Wheeler*, 1995; *Nordvedt*, 1998]

$$U = U_0 + U_{\mathrm{Y}} = -\frac{GMm}{r}\left(1 + \alpha e^{-\frac{r}{\lambda}}\right), \tag{1}$$

where $G$ is the Newtonian gravitational constant, $M$ is the mass of the central body, $m$ is the mass of the orbiting test particle, $r$ is the distance between the two bodies, $\alpha = \frac{Kk}{GMm}$ contains the couplings $K$ and $k$ of the new force to the two bodies[1] and $\lambda$ is the finite range of the new force.

In checking the nature of the fifth force it is of the utmost importance to perform experiments spanning the widest range of length scales as possible [*Nordvedt*, 1998]: for experiments at laboratory scale see [*Krause and Fischbach*, 2001] and references therein. Using the orbits of LAGEOS and LAGEOS II satellites implies that we are testing the hypothesis of the fifth force at a length scale of almost two Earth radii, i.e. $10^4$ km: at this scale the constraints on $\alpha$ are of the order of $|\alpha| < 10^{-5} - 10^{-8}$ (see Fig. 3.2 (a) of [*Ciufolini and Wheeler*, 1995]) and are derived from a data analysis of LAGEOS.

Our analysis includes also an evaluation of the error budget in order to account for various systematic errors induced by several classical aliasing forces. We will show that it is possible to improve sensibly the present limits by using suitably the data from the existing LAGEOS and LAGEOS II satellites and the present or near future knowledge of the terrestrial gravitational field whose uncertainties represent, as we will see later, the main sources of systematical errors.

The paper is organized as follows. In section 2 we derive the effects of the Yukawa gravitational potential on the orbit of a test body with the standard technique of the Gauss perturbative equations for the rates of change of the Keplerian orbital elements. In section 3 we apply the results obtained in section 2 to the Earth–LAGEOS system and discuss the constraints posed on $\alpha$ by using a suitable combination of the residuals of the perigee of LAGEOS II and

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[1]In general, $K$ and $k$ are not proportional to $M$ and $m$.

the nodes of LAGEOS II and LAGEOS. The effects of various sources of systematical errors are investigated. The role of the proposed LARES satellite is considered as well. Section 4 is devoted to the conclusions.

## 2 The orbital effects of the Yukawa perturbation

The acceleration felt by a test body orbiting the mass $M$ in the potential energy given by eq.(1) is

$$\mathbf{a} = \mathbf{a}_0 + \mathbf{a}_{\text{pert}} = -\frac{GM}{r^2}\hat{r} - \alpha GM\left(\frac{1}{r^2} - \frac{1}{\lambda^2}\right)\hat{r}. \tag{2}$$

In obtaining eq.(2) it has been assumed that $e^{-\frac{r}{\lambda}} \sim 1 - \frac{r}{\lambda}$, as it should be the case for an Earth orbiting satellite with $r \sim 10^7$ m. In eq.(2) $\hat{r}$ is the unit vector pointing from the central mass to the orbiting body.

The second term of the right–hand side of eq.(2) can be considered a small, central perturbation of the Newtonian monopole acceleration. It may be interesting to note that, since, in general, $\alpha$ may vary from a body to another in the field of the same mass[2], eq.(2) implies that different bodies may be accelerated differently in the field of $M$ [*Nordvedt*, 1998].

Let us work out explicitly the effects of the Yukawa perturbing acceleration on the orbit of an artificial satellite. We will adopt the standard approach based on the Gauss perturbative equations and the projections of the disturbing acceleration $R$, $T$, $N$ onto the radial, along–track and cross–track mutually orthogonal directions [*Milani et al.*, 1987], respectively. The Gauss equations are

$$\frac{da}{dt} = \frac{2}{n\sqrt{1-e^2}}\left[Re\sin f + T\frac{p}{r}\right], \tag{3}$$

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{na}\left[R\sin f + T\left(\cos f + \frac{1}{e}\left(1 - \frac{r}{a}\right)\right)\right], \tag{4}$$

$$\frac{di}{dt} = \frac{1}{na\sqrt{1-e^2}}N\frac{r}{a}\cos(\omega + f), \tag{5}$$

$$\frac{d\Omega}{dt} = \frac{1}{na\sin i\sqrt{1-e^2}}N\frac{r}{a}\sin(\omega + f), \tag{6}$$

$$\frac{d\omega}{dt} = -\cos i\frac{d\Omega}{dt} + \frac{\sqrt{1-e^2}}{nae}\left[-R\cos f + T\left(1 + \frac{r}{p}\right)\sin f\right], \tag{7}$$

[2]This happens if $k$ is not proportional to $m$.

$$\frac{d\mathcal{M}}{dt} = n - \frac{2}{na}R\frac{r}{a} - \sqrt{1-e^2}\left(\frac{d\omega}{dt} + \cos i \frac{d\Omega}{dt}\right), \tag{8}$$

where $a$, $e$, $i$, $\Omega$, $\omega$ and $\mathcal{M}$ are the satellite's semimajor axis, eccentricity, inclination, longitude of the ascending node, argument of perigee and mean anomaly, respectively. Moreover, $p = a(1-e^2)$, $f$ is the true anomaly and $n = \sqrt{GMa^{-3}}$ is the Keplerian mean motion. As can be noticed from eq.(2), the Yukawa disturbing acceleration has only the in–plane, radial component $R$, so that it can be straightforwardly inferred from eqs.(5)-(6) that the out–of–plane Keplerian orbital elements like the inclination $i$ and the longitude of the ascending node $\Omega$ are not affected by it. By evaluating the Yukawa acceleration on the unperturbed Keplerian ellipse, for which

$$r = \frac{a(1-e^2)}{1+e\cos f}, \tag{9}$$

inserting it in eqs.(3)-(8) and, then, averaging them over an orbital revolution, the long–period rates of change of the Keplerian orbital elements can be obtained. It turns out that, by neglecting terms of order $\mathcal{O}(e^n)$, with $n \geq 2$, in the satellite's eccentricity, only the perigee $\omega$ and the mean anomaly $\mathcal{M}$ are affected by long–term Yukawa perturbations. Indeed, from eq.(2) and eqs.(7)-(8) it can be obtained

$$\frac{d\omega}{dt} = \alpha\frac{n}{(1-e^2)^{\frac{3}{2}}}, \tag{10}$$

$$\frac{d\mathcal{M}}{dt} = n + \alpha\frac{n}{(1-e^2)} - \alpha\frac{2GM(1-e^2)}{na\lambda^2}. \tag{11}$$

## 3 The constraint on $\alpha$

In order to constraint effectively the Yukawa coupling $\alpha$, let us focus on eq.(10). It tells us that the Yukawa perturbation induces on the perigee of a near Earth satellite a secular rate which, for LAGEOS II, is proportional to $3.06658517 \times 10^{12}$ milliarcseconds per year (mas/y) via the coupling constant $\alpha$. What is the constraint posed on $\alpha$ by the experimental accuracy with which it can be possible to measure the perigee rate? In the case of this Keplerian orbital element the observable quantity is $r = ea\omega$. So, by assuming an experimental error of, say, $\delta r_{\text{exp}}^{\text{II}} = 1$ cm over 1 year, for LAGEOS II, which has $e = 0.014$ and $a = 1.2163 \times 10^9$ cm, we

have $\delta\omega_{\mathrm{exp}}^{\mathrm{II}} = 12$ mas[3]. This yields a relative accuracy on $\alpha$

$$\left(\frac{\delta\alpha}{\alpha}\right)^{\mathrm{exp}} \sim 4 \times 10^{-12}. \tag{12}$$

It is interesting to note that the same estimate for the proposed LARES satellite [*Ciufolini and Matzner*, 1998], which is planned to have a larger eccentricity, $e_{\mathrm{LARES}} = 0.04$, would yield a relative accuracy of almost $1 \times 10^{-12}$.

However, this estimate does not include any systematic errors. As it is well known from various proposed or performed tests of General Relativity with LAGEOS satellites [*Ciufolini et al.*, 1997; 1998; *Ciufolini*, 2000; *Iorio and Pavlis*, 2001; *Iorio*, 2001; *Iorio et al.*, 2002; *Iorio*, 2002], in such kind of measurements there are lots of competing classical forces which may act as superimposed biases affecting sensibly the precision of the measurements. Then, the evaluation of the systematical errors induced by them is of the utmost importance.

The main source of systematic errors is represented by the mismodelled precessions of the even zonal harmonics of the static part of the geopotential. In particular, the first two even zonal harmonics $J_2$ and $J_4$ are the most insidious. In order to cancel their impact, as proposed in the PPN LAGEOS experiment [*Iorio et al.*, 2002], the following combination of orbital residuals could be used[4]

$$\delta\dot{\omega}^{\mathrm{II}} + c_1\delta\dot{\Omega}^{\mathrm{II}} + c_1\delta\dot{\Omega}^{\mathrm{I}} = \alpha x_{\mathrm{Yuk}}, \tag{13}$$

where

$$c_1 = -0.86, \tag{14}$$

$$c_2 = -2.85, \tag{15}$$

$$x_{\mathrm{Yuk}} = 3.06658517 \times 10^{12} \text{ mas/y}. \tag{16}$$

The coefficients of eq.(13) depend on the orbital parameters of LAGEOS and LAGEOS II and are obtained in order to cancel the contributions of the first two even zonal harmonics of the

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[3]The other existing passive geodetic laser–ranged satellites are unsuitable because their eccentricities are smaller than LAGEOS II, except for Starlette whose orbit, however, is known less accurately than that of LAGEOS II for various reasons.

[4]Notice that, since LAGEOS enters the combination of eq.(13) only with its node $\Omega$, and since the Yukawa perturbation does not affect such Keplerian orbital element, eq.(13) is insensitive to the possible differential falling of LAGEOS and LAGEOS II in the terrestrial gravitational field.

geopotential to the measurement of $\alpha$. It is intended that the residuals would account for the Yukawa perturbation in the sense that it would be viewed as an unmodelled feature not included in the force models adopted in fitting the satellites' orbits. According to the covariance matrix of the most recent available Earth gravity model EGM96 [*Lemoine et al.*, 1998], the systematic error induced by the uncancelled even zonal harmonics of the geopotential amounts to[5]

$$\left(\frac{\delta\alpha}{\alpha}\right)^{\text{zonals}} \sim 7 \times 10^{-12}. \tag{17}$$

Regarding the time–dependent part of the Earth gravitational field, the estimates of [*Iorio et al.*, 2002], adapted to this context, yield for an observational time span of 5 years

$$\left(\frac{\delta\alpha}{\alpha}\right)^{\text{harmonics}} \sim 1 \times 10^{-12}. \tag{18}$$

The most relevant non–gravitational perturbations are the direct solar radiation pressure and the Earth's albedo. Their impact can be evaluated from [*Lucchesi*, 2001] for 5 years as

$$\left(\frac{\delta\alpha}{\alpha}\right)^{\text{non−grav}} \sim 8 \times 10^{-12}. \tag{19}$$

The errors induced by the direct solar radiation pressure and the Earth's albedo have been added quadratically, as suggested in [*Lucchesi*, 2001]. Regarding other subtle non–gravitational perturbations of thermal origin acting on the orbits of the LAGEOS satellites, their effects on the perigee of LAGEOS II are currently under accurate evaluation.

Then, a reliable estimate of the total systematic error induced by various classical perturbations on the measurement of $\alpha$ through eq.(13) over a 5 years time span yields

$$\left(\frac{\delta\alpha}{\alpha}\right)^{\text{total}} \sim 1 \times 10^{-11}. \tag{20}$$

In obtaining eq.(20) we have summed in a root–sum–square fashion the gravitational and non–gravitational errors assumed to be independent. Instead, the gravitational error due to the static and time–dependent parts of the Earth's gravitational field have been simply summed up in view of a reciprocal correlation.

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[5]This estimate has been obtained by considering the geopotential harmonics up to degree $l = 20$. This is well justified by the insensitivity of LAGEOS satellites to the higher degree terms. Moreover, this fact makes our estimate reliable because the higher degree terms of geopotential in EGM96 are not particularly well determined.

# 4  Conclusions

In this paper we have shown that by using a suitable combination of the orbital residuals of the perigee of LAGEOS II and the nodes of LAGEOS II and LAGEOS laser–ranged Earth satellites it would be possible to constrain effectively the Yukawa coupling $\alpha$ of a possible fifth force at a length scale of almost two Earth's radii. The experimental sensitivity in measuring the perigee, which would be affected by the Yukawa force, of LAGEOS II would allow a precision on $\alpha$ of the order of almost $4 \times 10^{-12}$ over a time span of 1 year. According to the present knowledge of the terrestrial gravity field, the constraint posed by the systematic errors is of the order of $1 \times 10^{-11}$ over an observational time span of 5 years. These estimates should greatly improve when the new and more accurate data on the Earth gravity field from the CHAMP and GRACE missions will be available in the near future. The use of the proposed LARES satellite, with its larger eccentricity, would allow to improve the experimental constraint to $1 \times 10^{-12}$.

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